# University of Puerto Rico at Rio Piedras
# Department of Mathematics
# Qualifying Exam
# MATE 6682: Algorithms

September 11, 2014

| **Your Name** | **Your Student ID** |
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**Instructions:** Solve **exactly three** of the following five exercises. Please read carefully, write clearly and show all your work to get credit. Good luck!

**Exercise 1 (100 points)**

(a) (60 points) For each of following pairs of functions, state which one grows faster and prove your results.

(i) $n^{\log n}$ or $(\log n)^n$

(ii) $n^2/\log n$ or $\log^2 n$

(iii) $n^{1/n}$ or $\log n$

(b) (40 points) In each of the following cases, find asymptotic upper and lower bounds for $T(n)$. Assume $T(n)$ is constant for $n \leq 2$. Make your bounds as tight as possible, and justify your answers.

(i) $T(n) = 2T(\frac{n}{2}) + n^4$. You may assume that $n$ is a power of 2.

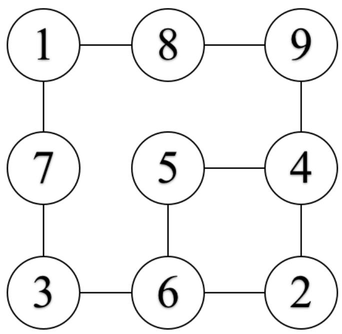
(ii) $T(n) = 4T(\frac{n}{3}) + n \log n$. You may assume that $n$ is a power of 3.

**Exercise 2 (100 points)**
Given undirected graph $G = (V, E)$ :

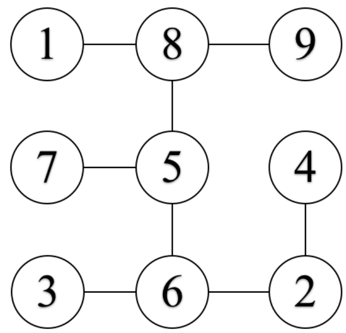
(a) (50 points) Describe an algorithm that determines whether or not $G$ contains a cycle.

(a) (20 points) Your algorithm should run in $O(V)$ time, independent of the number of edges.

(c) (30 points) Apply your algorithm to the graphs $G_1$ and $G_2$, given bellow.




Graph $G_1$




Graph $G_2$

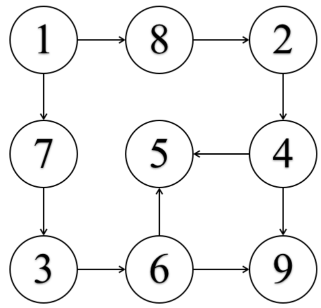
**Exercise 3 (100 points)**
In the following problem, $|V|$ denotes the number of elements in $V$ and $x \rightsquigarrow y$ denotes a path starting at vertex $x$ and ending at vertex $y$.

Let $G = (V, E)$ be a directed graph in which each vertex $u \in V$ is labeled with a unique integer $L(u)$ from the set $\{1, 2, \ldots, |V|\}$. For each vertex $u \in V$, let $R(u) = \{v \in V : u \rightsquigarrow v\}$ be the set of vertices that are reachable from $u$. Define $\min(u)$ to be the vertex in $R(u)$ whose label is minimum, i.e., $\min(u)$ is the vertex $v$ such that $L(v) = \min\{L(w) : w \in R(u)\}$.

(a) (60 points) Give an $O(V + E)$–time algorithm that computes $\min(u)$ for all vertices $u \in V$. Prove that your running time is indeed $O(V+E)$.

(b) (40 points) Apply your algorithm to the graph $G$ given bellow.

*Note that:* (1) every vertex is reachable from itself and (2) you don't have to compute $R(u)$ for any $u \in V$ in order to find $\min(u)$.




Graph $G$

**Exercise 4 (100 points)**
In the $20 \times 20$ matrix bellow, four numbers along a diagonal have been marked in red.

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 08 | 02 | 22 | 97 | 38 | 15 | 00 | 40 | 00 | 75 | 04 | 05 | 07 | 78 | 52 | 12 | 50 | 77 | 91 | 08 |
| 49 | 49 | 99 | 40 | 17 | 81 | 18 | 57 | 60 | 87 | 17 | 40 | 98 | 43 | 69 | 48 | 04 | 56 | 62 | 00 |
| 81 | 49 | 31 | 73 | 55 | 79 | 14 | 29 | 93 | 71 | 40 | 67 | 53 | 88 | 30 | 03 | 49 | 13 | 36 | 65 |
| 52 | 70 | 95 | 23 | 04 | 60 | 11 | 42 | 69 | 24 | 68 | 56 | 01 | 32 | 56 | 71 | 37 | 02 | 36 | 91 |
| 22 | 31 | 16 | 71 | 51 | 67 | 63 | 89 | 41 | 92 | 36 | 54 | 22 | 40 | 40 | 28 | 66 | 33 | 13 | 80 |
| 24 | 47 | 32 | 60 | 99 | 03 | 45 | 02 | 44 | 75 | 33 | 53 | 78 | 36 | 84 | 20 | 35 | 17 | 12 | 50 |
| 32 | 98 | 81 | 28 | 64 | 23 | 67 | 10 | **26** | 38 | 40 | 67 | 59 | 54 | 70 | 66 | 18 | 38 | 64 | 70 |
| 67 | 26 | 20 | 68 | 02 | 62 | 12 | 20 | 95 | **63** | 94 | 39 | 63 | 08 | 40 | 91 | 66 | 49 | 94 | 21 |
| 24 | 55 | 58 | 05 | 66 | 73 | 99 | 26 | 97 | 17 | **78** | 78 | 96 | 83 | 14 | 88 | 34 | 89 | 63 | 72 |
| 21 | 36 | 23 | 09 | 75 | 00 | 76 | 44 | 20 | 45 | 35 | **14** | 00 | 61 | 33 | 97 | 34 | 31 | 33 | 95 |
| 78 | 17 | 53 | 28 | 22 | 75 | 31 | 67 | 15 | 94 | 03 | 80 | 04 | 62 | 16 | 14 | 09 | 53 | 56 | 92 |
| 16 | 39 | 05 | 42 | 96 | 35 | 31 | 47 | 55 | 58 | 88 | 24 | 00 | 17 | 54 | 24 | 36 | 29 | 85 | 57 |
| 86 | 56 | 00 | 48 | 35 | 71 | 89 | 07 | 05 | 44 | 44 | 37 | 44 | 60 | 21 | 58 | 51 | 54 | 17 | 58 |
| 19 | 80 | 81 | 68 | 05 | 94 | 47 | 69 | 28 | 73 | 92 | 13 | 86 | 52 | 17 | 77 | 04 | 89 | 55 | 40 |
| 04 | 52 | 08 | 83 | 97 | 35 | 99 | 16 | 07 | 97 | 57 | 32 | 16 | 26 | 26 | 79 | 33 | 27 | 98 | 66 |
| 88 | 36 | 68 | 87 | 57 | 62 | 20 | 72 | 03 | 46 | 33 | 67 | 46 | 55 | 12 | 32 | 63 | 93 | 53 | 69 |
| 04 | 42 | 16 | 73 | 38 | 25 | 39 | 11 | 24 | 94 | 72 | 18 | 08 | 46 | 29 | 32 | 40 | 62 | 76 | 36 |
| 20 | 69 | 36 | 41 | 72 | 30 | 23 | 88 | 34 | 62 | 99 | 69 | 82 | 67 | 59 | 85 | 74 | 04 | 36 | 16 |
| 20 | 73 | 35 | 29 | 78 | 31 | 90 | 01 | 74 | 31 | 49 | 71 | 48 | 86 | 81 | 16 | 23 | 57 | 05 | 54 |
| 01 | 70 | 54 | 71 | 83 | 51 | 54 | 69 | 16 | 92 | 33 | 48 | 61 | 43 | 52 | 01 | 89 | 19 | 67 | 48 |

The product of these numbers is $26 \times 63 \times 78 \times 14 = 1788696$.

(a) (70 points) Provide an algorithm to compute the greatest product of four adjacent numbers in the same direction (up, down, left, right, and diagonally) in an $n \times n$ matrix.

(b) (30 points) Analyze the time complexity of your solution.

**Exercise 5 (100 points)**
Write a program which takes as input two integer numbers, $n \geq 3$ and $c \geq 1$, and returns the value of the function $f(n)$ defined as follows:

$$f(n) = 4f(\lfloor n/3 \rfloor) + \lfloor n \log_2 n \rfloor,$$

where $f(1) = c$. Here, the function $\lfloor x \rfloor$ returns the greatest integer $t$ for which $t \leq x$. What is asymptotically the running time of your program?